March 2001 (revised May 2001)

PAR-LPTHE 01/15
Ref. SISSA 28/2001/EP

HIERARCHIES OF QUARK MASSES AND THE MIXING MATRIX IN THE STANDARD THEORY

B. Machet [1] [2] & S.T. Petcov [3] [4]

Abstract. We study the general dependence of mixing angles on heavy fermion masses when mass hierarchies exist among the fermions. For two generations and small Cabibbo angle, this angle is directly shown to scale like $\mu_1/m_s \pm \mu_2/m_c$, where $|\mu_1| \ll m_s, |\mu_2| \ll m_c$ are independent mass scales. For $n = 3$ generations, we extend to the Yukawa matrices of u- and d-type quarks the property that the 2×2 upper-left sub-matrix of the Cabibbo-Kobayashi-Maskawa matrix K is a good approximation to the Cabibbo matrix C. Then, without any additional Ansatz concerning the existence of mass hierarchies or the smallness of the mixing angles, the moduli of its entries $K_{13}, K_{23}, K_{31}, K_{32}$ are shown to scale like $[\beta_{13}, \beta_{23}, \beta_{31}, \beta_{32}]\sqrt{m_c/m_t} \pm [\delta_{13}, \delta_{23}, \delta_{31}, \delta_{32}]\sqrt{m_s/m_b}$, where the β's and the δ's are coefficients smaller than 10. This method, when used for two generations, gives a dependence on m_s and m_c "weaker" than the one obtained first, but which matches a well known behaviour for the Cabibbo angle: $\theta_c \approx \sqrt{\epsilon_d(m_d/m_s)} - \sqrt{\epsilon_u(m_u/m_c)}$, with $\epsilon_d, \epsilon_u \leq 1$. The asymptotic behaviour in the case of three generations can also be strengthened into a $1/m_{b,t}$ behaviour by incorporating our knowledge about the hierarchies of quark masses and the smallness of the mixing angles.

PACS: 12.15.Ff

CENTRE NATIONAL
DE LA RECHERCHE
SCIENTIFIQUE

[1]LPTHE, tour 16 / 1er étage, Université P. et M. Curie, BP 126, 4 place Jussieu, F-75252 Paris Cedex 05 (France).
Unité associée au CNRS et aux Universités Paris 6 (P. et M. Curie) et Paris 7 (Denis Diderot): UMR 7589

[2]Email: machet@lpthe.jussieu.fr

[3]SISSA, Via Beirut 2-4, I-34013 Trieste (Italy)

[4]E-mail: petcov@he.sissa.it

1 Introduction

The fermionic mass spectrum and the mixing angles are intimately connected in the Standard Model [1]. However, the number of independent (Yukawa) couplings being larger than the total number of masses and mixing angles, one cannot deduce, without additional assumptions concerning flavour physics beyond the Standard Model, one-to-one relationships between mixing angles and (ratios of) fermion masses. In general, any relation between them involves arbitrary entries of the Yukawa matrices; renormalization group arguments [2][3][4] add to this indetermination by showing that the latter can undergo large variations when the largest mass scale runs from the electroweak scale to the unification scale. The presence of a number of zeroes in the Yukawa matrices at the unification scale can eventually reduce the number of independent variables to the one of measurable quantities, and this possibility was extensively studied [3][5][6][7] [8] [4] [9][10][11][12]. The missing information could also be supplied by introducing additional flavour symmetries into the Standard Model or its extensions (see for example [13] and references therein).

In the present article, we derive general "asymptotic" relations between mixing angles and quark masses, valid, strictly speaking, when the two heaviest quarks of the u- and d- type have masses much larger than the others. These relations are obtained from the following requirements, the first being dictated by experiment:
- there exist hierarchies between the quark masses inside each family;
- one must be able to describe correctly, *i.e.* accurately and coherently, physics with a given number of observed families, even if there exist additional families.

In the case of two generations, taking into account the existence of separate hierarchies between the masses of the two charge $2/3$ quarks and the masses of the two charge $-1/3$ quarks, and the fact that the Cabibbo angle is small, we show that the latter behaves like $\theta_c \approx \mu_1/m_s \pm \mu_2/m_c$ when $m_s \gg m_d$ and $m_c \gg m_u$, where m_s, m_d, m_c, m_u are respectively the masses of the s, d, c and u quarks, $\mu_{1,2}$ are real independent mass scales and $|\mu_1| \ll m_s, |\mu_2| \ll m_c$.

As the method used for $n = 2$ generations cannot be straightforwardly extended to a larger number of families, for $n = 3$, we do not make any hypothesis concerning the existence of hierarchies or the smallness of the mixing angles but, instead, we introduce another requirement, shown to be very mild: like the 2×2 upper-left sub-matrix of the Cabibbo-Kobayashi-Maskawa (CKM) matrix K [14] matches the Cabibbo matrix to a very good approximation, we require that, for each (u and d) type of quarks, taking as Yukawa mass matrix for two generations the upper-left 2×2 sub-matrix of the one for three generations yields an uncertainty in the mass spectrum (of two generations) which is much smaller than the larger of the two quark masses. This condition yields by itself interesting constraints on the asymptotic behaviour of mixing angles. For two generations, one recovers the well-known result that the Cabibbo angle behaves, when $m_s \gg m_d, m_c \gg m_u$ like $\sqrt{\epsilon_d(m_d/m_s)} - \sqrt{\epsilon_u(m_u/m_c)}$, where $\epsilon_d, \epsilon_u \leq 1$. For $n = 3$, a similar behaviour is demonstrated for the elements $K_{13}, K_{31}, K_{23}, K_{32}$ of the CKM mixing matrix.

The above "stability" of the mixing matrix as one goes from n to $(n\pm1)$ generations could be fortuitous: nature may be such that the whole structure of mass matrices for $(n-1)$ generations is totally spoiled when one goes to n [1] ; one then would probably face the fact that, if there exist generations beyond the ones presently known, the knowledge about Yukawa matrices established at our energies is mostly illusory and, from it, we cannot expect reliable hints for the "complete" theory. In particular, this would spoil the renormalization group arguments which evolve Yukawa matrices to the unification scale [2][3][4], where one can expect higher generations. As can be easily seen, "stable" Yukawa mass matrices yield "stable" mixing matrices; we go further and demand here that the "stability" of the Yukawa matrices be a necessary condition for the one of the mixing matrix.

Adding the information that there exist hierarchies for the masses of the u- and d- type quarks and that

[1] As this is evidently untrue for their diagonal form, this would tend to emphasize the role of non-diagonal elements.

the mixing angles are small, as represented in a Wolfenstein-like parameterization [15] of the CKM mixing matrix, this behaviour can be strengthened into a $1/m_{b,t}$ behaviour, like for two generations.

2 Basic properties of Yukawa matrices in the Standard Model

The following well known properties concerning the Standard Model will be extensively used in the paper. Let M_u and M_d be the Yukawa mass matrices for the u- and d-type quarks.

• *(i) A priori*, M_u and M_d have no special property of symmetry, and their eigenvalues (defined as the roots of their characteristic equations) are not the quark masses; the latter are determined by diagonalizing $MM^\dagger$ and $M^\dagger M$, which are hermitian, by unitary U and V matrices, according to:

$$U_u^\dagger M_u M_u^\dagger U_u = D_u^2, \quad V_u^\dagger M_u^\dagger M_u V_u = D_u^2, \quad U_d^\dagger M_d M_d^\dagger U_d = D_d^2, \quad V_d^\dagger M_d^\dagger M_d V_d = D_d^2. \quad (1)$$

The entries of the diagonal matrices D_u and D_d are the quark masses. Their signs are irrelevant. M_u and M_d are brought to their diagonal forms by bi-unitary transformations

$$U_u^\dagger M_u V_u = D_u, \quad U_d^\dagger M_d V_d = D_d, \quad (2)$$

corresponding to acting on left-handed fermions with the U's and on right-handed ones with the V's. The CKM mixing matrix is then [2]

$$K = U_u^\dagger U_d. \quad (3)$$

The unitarity of U and V ensures that the kinetic terms keep diagonal.

• *(ii)* Arbitrary independent rotations can be performed on u- and d-type right-handed quarks (*i.e.* quark fields), and identical transformations on left-handed u- and d-type quarks.

• *(iii)* One can swap columns in M by a rotation on right-handed quarks (equivalent to right-multiplying $M \to MR$); one can swap lines by a rotation on left-handed quarks, but the same rotation must be done on u- and d-type quarks ($M_u \to LM_u, M_d \to LM_d$).

• *(iv)* Any mass matrix can be brought to a triangular form by rotations on right-handed quarks.

• *(v) Polar decomposition theorem*[16] [17]: any complex matrix M can be written like the product of an hermitian matrix $(MM^\dagger)^{1/2}$ times a unitary matrix; in the Standard Model, the latter can always be absorbed by a rotation on right-handed quarks; hence, there, all mass matrices can always be taken to be hermitian; one has then $U = V$, and the quark masses become identical to the eigenvalues of the mass matrices.

• *(vi)* K (3) is left invariant if U_u and U_d are multiplied by the same unitary matrix A; this is also true for the corresponding right-handed mixing matrix $K_R = V_u^\dagger V_d$ (of no relevance in the Standard Model) when V_u and V_d are multiplied by A; thus M_u and M_d are determined up to a common unitary transformation $M_u \to A^\dagger M_u A, M_d \to A^\dagger M_d A$.

3 The case of two generations

We investigate here the relation between the Cabibbo angle θ_c and the quark masses. The case of two generations is considered separately and can be treated in a simple way which only supposes the existence of hierarchies among each type of quarks and uses the observed smallness of the mixing angle. The Yukawa matrices can be taken to be real, which we assume here.

First, we exploit the existence of mass hierarchies for the u- and d-type quarks, *i.e.* that $m_s \gg m_d$ and $m_c \gg m_u$. For each type of quarks, this hierarchy of masses requires that at least one of the entries in the corresponding Yukawa matrix M must be larger than – or equal to – the larger of the two quark masses; otherwise all eigenvalues of MM^T and M^TM [3] would be smaller than the scale Λ set by the

[2] A trivial consequence of (3) is that, in a basis where M_u is diagonal ($U_u = \mathbb{I} = V_u$), $K = U_d$.

[3] The superscript "T" means "transposed".

larger quark mass. M can have one, two, three or four entries larger or equal to Λ. The case of four large entries can always be reduced to three by a right-handed rotation which, for example, brings M to the upper triangular form. If there are three large entries, the sum and the product of the eigenvalues of MM^T are large, which mean that both quark masses are larger than Λ.

The case with two large entries can be further reduced in that they cannot be both on the diagonal, since, then, the two quark masses are again larger than Λ. Furthermore, they must be on the same column. Suppose indeed that they are on the same line. A right-handed rotation can then always reduce the number of large entries to one. For instance [4]

$$MR = \begin{pmatrix} \Lambda_1 & \Lambda_2 \\ a & b \end{pmatrix} \begin{pmatrix} c_\varphi & s_\varphi \\ -s_\varphi & c_\varphi \end{pmatrix} \to c_\varphi \begin{pmatrix} \Lambda_1 + \Lambda_2^2/\Lambda_1 & 0 \\ a + b\Lambda_2/\Lambda_1 & -a\Lambda_2/\Lambda_1 + b \end{pmatrix} \quad \text{for } s_\varphi = -c_\varphi \Lambda_2/\Lambda_1 \tag{4}$$

where $|\Lambda_{1,2}| \geq \Lambda$ and $|a|, |b| \ll \Lambda$, has only one large entry at the $[1,1]$ position. The same argument applies when the two initial large entries are on the second line.

Suppose, accordingly, that the two large entries are on the same column, and use the freedom to make common left-handed rotations on both u and d type quarks:

$$LM = \begin{pmatrix} c_\varphi & s_\varphi \\ -s_\varphi & c_\varphi \end{pmatrix} \begin{pmatrix} \Lambda_1 & a \\ \Lambda_2 & b \end{pmatrix} \to c_\varphi \begin{pmatrix} \Lambda_1 + \Lambda_2^2/\Lambda_1 & a + b\Lambda_2/\Lambda_1 \\ 0 & -a\Lambda_2/\Lambda_1 + b \end{pmatrix} \quad \text{for } s_\varphi = c_\varphi \Lambda_2/\Lambda_1; \tag{5}$$

LM has now only one large entry at the $[1,1]$ position and this rotation does not alter the structure of the second mass matrix, supposed to also have two large entries on the same column (whatever be the column). So we can state:

For two generations, the existence, for each type of quarks, of one mass larger than a certain scale, the other being much smaller than this scale, entails that one can restrict to Yukawa matrices having either one large entry each, or to the case where one of them has two large entries in the same column and the other has only one large entry; by a right-handed rotation, the latter can be put in any column.

• Accordingly, we consider, first, for example, M_u with two large entries on the same column and M_d with only one

$$M_u = \begin{pmatrix} \Lambda_1 & a \\ \Lambda_2 & b \end{pmatrix}, \quad |\Lambda_1|, |\Lambda_2| \gg |a|, |b|, \quad M_d = \begin{pmatrix} u & v \\ w & \Lambda_3 \end{pmatrix}, \quad |\Lambda_3| \gg |u|, |v|, |w|. \tag{6}$$

The successive steps are the following. From (3), the Cabibbo angle can be written as

$$\theta_c = \theta_d - \theta_u. \tag{7}$$

Using (6) and diagonalizing $M_d M_d^T$ we get

$$\tan(2\theta_d) = 2\frac{uw + v\Lambda_3}{\Lambda_3^2 + w^2 - u^2 - v^2}. \tag{8}$$

Thus, when $|\Lambda_3| \gg |u|, |v|, |w|$, θ_d is small and behaves like $\theta_d \approx v/\Lambda_3$. From (6) one finds similarly

$$\tan(2\theta_u) = 2\frac{ab + \Lambda_1\Lambda_2}{\Lambda_2^2 - \Lambda_1^2 + b^2 - a^2}. \tag{9}$$

[4]Throughout the paper the use the abbreviated notations c for $\cos$ and s for $\sin$. The subscripts referring to the corresponding angles should avoid any confusion with the the charm and strange quarks.

We now use as input our knowledge about the Cabibbo angle: as θ_c is experimentally small and θ_d has just been proven to be small, (7) requires that θ_u is also small. This can only occur if $|\Lambda_1| \gg |\Lambda_2|$ or $|\Lambda_2| \gg |\Lambda_1|$. Let us consider, for example, $|\Lambda_1| \gg |\Lambda_2|$. Then θ_u behaves like $-\Lambda_2/\Lambda_1$. Thus, θ_c behaves like $v/\Lambda_3 + \Lambda_2/\Lambda_1$. The quark masses are

$$|m_d| \approx |u|, \quad |m_s| \approx |\Lambda_3|, \quad |m_u| \approx \frac{|a\Lambda_2 - b\Lambda_1|}{\sqrt{\Lambda_1^2+\Lambda_2^2}} \approx |b|, \quad |m_c| \approx \sqrt{\Lambda_1^2+\Lambda_2^2} \approx |\Lambda_1|. \tag{10}$$

The limits $m_s \gg m_d, m_c \gg m_u$ do not depend on the non-diagonal elements a, Λ_2 v, w of the mass matrices, which stand, in this limit, as independent entries only determined by physics of flavour beyond the Standard Model. This allows the claim: given the fact that $m_s \gg m_d, m_c \gg m_u$,

$$\theta_c \approx \frac{v}{m_s} \pm \frac{\Lambda_2}{m_c}, \quad m_s \gg |v|, m_c \gg |\Lambda_2|. \tag{11}$$

- The case when the two mass matrices M_u and M_d have only one large entry each is trivial since, then, one also gets a small θ_u and the equivalent of (8) for $\tan(2\theta_u)$.
- One can consequently state the general result for the Cabibbo angle:

$$\theta_c \approx \frac{\mu_1}{m_s} \pm \frac{\mu_2}{m_c}, \quad m_s \gg |\mu_1|, m_c \gg |\mu_2|, \tag{12}$$

where $\mu_{1,2}$ are independent real mass scales and we have neglected corrections in higher inverse powers of m_s and m_c. The mass scales $\mu_{1,2}$ depend on the basis. If one goes to a basis where, e.g., M_u is diagonal, then $\theta_u = 0$ and $\mu_2 = 0$. This reflects the similar roles held by the two members inside an $SU(2)$ doublet and, in particular, that one cannot push one of the two masses alone to infinity without breaking the renormalizability of the theory [5].

4 Defining hierarchical matrices; algebraic properties

Consider the following set $\{\mathcal{M}\}_h$ of real $n \times n$ matrices $\mathcal{M}$ (n is here the number of generations of fermions) such that, for any of them:
- the modulus of its diagonal element $\mathcal{M}_{nn}$ is of order unity $|\mathcal{M}_{nn}| \approx 1$:
- its non-diagonal border (*i.e.* in the lowest line and extreme right column) elements are small in the following sense: $\mathcal{M}_{in} = \lambda_i/\Lambda_i, \mathcal{M}_{nj} = \xi_j/\Xi_j, i,j = 1\cdots(n-1), |\lambda_i| \ll |\Lambda_i|, |\xi_j| \ll |\Xi_j|$;
- its other elements $\mathcal{M}_{ij}, i,j = 1\cdots(n-1)$ satisfy $|\mathcal{M}_{ij}|_{i,j=1\cdots(n-1)} \leq 1$.

We shall call hereafter such matrices *hierarchical* [6].

Property 1: Neglecting terms of second order in λ_i/Λ_i and ξ_j/Ξ_j, and considering the usual multiplication of matrices:
- the product of two hierarchical matrices is hierarchical;
- the multiplication of hierarchical matrices is of course associative;
- the unit matrix, which is hierarchical, is the identity element;
- the inverse of a hierarchical matrix is also hierarchical.

$\{\mathcal{M}\}_h$ would form a group, but for the stability which is not ensured for the product of a very large number of hierarchical matrices.

The CKM mixing matrix K being given by (3) the properties just stressed entail that if U_u and U_d are hierarchical, then so is K.

Property 2: *in the same limit of neglecting terms of second order in λ_i/Λ_i and ξ_j/Ξ_j, the $(n-1) \times (n-1)$ sub-matrix $\tilde{\mathcal{M}}$ with entries $\tilde{\mathcal{M}}_{ij} = \mathcal{M}_{ij}, i = 1\cdots(n-1), j = 1\cdots(n-1)$ of a hierarchical*

[5] One has then to implement a non-linear realization of the gauge symmetry [18]

[6] This definition constrains only the non-diagonal elements on the border to be small.

$n \times n$ matrix $\mathcal{M}$ which is the product of two $n \times n$ hierarchical matrices $\mathcal{U}$ and $\mathcal{V}$ only depends of the two corresponding $(n-1) \times (n-1)$ sub-matrices $\tilde{\mathcal{U}}$ and $\tilde{\mathcal{V}}$ of $\mathcal{U}$ and $\mathcal{V}$.

A consequence is that the hierarchical structure of the Cabibbo matrix C can be studied without reference to the third generation after the CKM matrix has been proven to be itself hierarchical; explicitly, K is first proven to be hierarchical from the property ("upper" hierarchy) $m_b \gg m_s, m_d, m_t \gg m_c, m_u$ by proving that U_u and U_d (suitably normalized) are both hierarchical; then C can be proven to be hierarchical from the property ("lower" hierarchy) $m_s \gg m_d, m_c \gg m_u$, without worrying anymore about the third generation.

Property 3: *any unitary matrix U can be made hierarchical by left- or right- multiplication*. Indeed, for example, the equation $UA = U_h$ with U_h hierarchical, has always a solution $A = U^\dagger U_h$.
A consequence is that:

Property 4: *in the Standard Model and when the mixing matrix K is hierarchical, one can always go to a basis where U_u, U_d and the two (hermitian) mass matrices M_u, M_d, normalized to the largest scale available (see below), are hierarchical*.

Let us first go in a basis where M_u and M_d are hermitian, which is always possible by the polar decomposition theorem (property *(v)* of section 2). Then, we multiply each of the two unitary matrices U_u and U_d by the same unitary matrix B, which, as stressed in section 2 (property *(vi)*), leaves the mixing matrix K invariant. One chooses B such that $BU_u = U_{uh}$ is hierarchical, which is always possible as mentioned above; $BU_d = U_{dh}$ is then also hierarchical since $U_{dh} = KU_{uh}$, K has been supposed to be hierarchical, and hierarchical matrices are stable by multiplication.

These transformations on U_u and U_d are equivalent to a change of basis for quarks $M_u \to BM_uB^\dagger$, $M_d \to BM_dB^\dagger$ since the diagonalisation equations for M_u and M_d write $D_u = U_u^\dagger M_u U_u = U_{uh}^\dagger (BM_uB^\dagger) U_{uh}$ and $D_d = U_d^\dagger M_d U_d = U_{dh}^\dagger (BM_dB^\dagger) U_{dh}$; let us go to the mass matrices normalized to the largest entry m_{un} of D_u and m_{dn} of D_d: $\mathcal{D}_u = D_u/m_{un}$, $\mathcal{M}_{uh} = U_{uh}^\dagger (BM_uB^\dagger) U_{uh}/m_{un}$, $\mathcal{D}_d = D_d/m_{dn}$, $\mathcal{M}_{dh} = U_{dh}^\dagger (BM_dB^\dagger) U_{dh}/m_{dn}$; from $\mathcal{D}_u = U_{uh}^\dagger \mathcal{M}_{uh} U_{uh}$, $\mathcal{D}_d = U_{dh}^\dagger \mathcal{M}_{dh} U_{dh}$, from the fact that $\mathcal{D}_u$ and $\mathcal{D}_d$ are hierarchical, and from the stability property of hierarchical matrices, one deduces that $\mathcal{M}_{uh}$ and $\mathcal{M}_{dh}$ are hierarchical.

5 The case of $n \geq 2$ generations

A direct demonstration like the one of section 3, which uses the smallness of the mixing angles, needs, to be extended to n generations, inspecting all possible configurations of $n \times n$ mass matrices which lead to mass hierarchies between fermions. If feasible, it would become extremely tedious and inelegant. This is why we perform the generalization to $n = 3$ generations along another line which does not require any hypothesis concerning the properties of the mixing matrix. It provides, in the $n = 2$ case, a well-known constraint on θ_c, though weaker than the one deduced in section 3.

5.1 General requirement for Yukawa matrices

In the following, like in section 4, $\tilde{M}$ stands for the $(n-1) \times (n-1)$ upper left sub-block of any given $n \times n$ matrix M. One takes M hermitian (see section 2); $\tilde{M}$ is then also hermitian. A single unitary matrix U is needed to diagonalize M into D.

If one steps up or down one generation from n to $(n+1)$ or $(n-1)$, one must be able to find a renormalizable description of the corresponding physics: for example, before the third generation was suspected, and discovered, the Cabibbo description of two generations was perfectly coherent, and our present description of physics with three generations, also coherent, is supposed to be very little influenced by the eventual presence of higher generations.

Keeping the previous notations for n generations, let us step down to $(n-1)$ generations. According to our requirement, there exist, in this case, too, Yukawa matrices $\underline{M}$, which can also be taken to be hermitian, and the unitary matrices which diagonalize them into $\underline{D}$ are called $\underline{U}$.

In general, $\underline{M} \neq \tilde{M}, \underline{U} \neq \tilde{U}$. Experimentally, however, the (unitary) Cabibbo matrix $C \equiv \underline{U_u^T U_d}$ is very close to the upper 2×2 (non-unitary) sub-matrix $\tilde{K} = \widetilde{U_u^T U_d}$ of the CKM matrix K.

As proposed, we demand that this property of "stability" of the mixing matrix be the image of an equivalent property for the Yukawa matrices:

$$\underline{M} = \tilde{M} - \epsilon\, m_{n-1}\mathbb{I}_{n-1}, \tag{13}$$

where:

- m_n and m_{n-1} are the largest quark masses of the type considered (u or d), respectively for n and $(n-1)$ generations; for instance $m_n = m_b\ or\ m_t$ and $m_{n-1} = m_s\ or\ m_c$ for $n=3$; $m_n = m_s\ or\ m_c$ and $m_{n-1} = m_d\ or\ m_u$ for $n=2$;
- ϵ, which can *a priori* depend on the entries of $\tilde{M}$, is small in the sense $\epsilon| \leq 1$;
- $\mathbb{I}_{n-1}$ is any $(n-1)\times(n-1)$ matrix with entries of moduli not larger than one, for example the $(n-1)\times(n-1)$ identity matrix [7] .

Expanding

$$\tilde{M} \equiv \widetilde{UDU^\dagger} = \tilde{U}\underline{D}\tilde{U}^\dagger + m_n\tilde{A}, \tag{14}$$

$\tilde{A}$ is an $(n-1)\times(n-1)$ matrix which contains the elements of the mixing matrix for which we are seeking information; one gets from (13) the condition

$$\underline{M} \equiv \underline{UDU^\dagger} = \tilde{U}\underline{D}\tilde{U}^\dagger + m_n\tilde{A} - \epsilon\, m_{n-1}\mathbb{I}_{n-1}, \quad |\epsilon| \leq 1, \quad or \tag{15}$$

$$\tilde{A} = \frac{\underline{M} - \tilde{U}\underline{D}\tilde{U}^\dagger + \epsilon\, m_{n-1}\mathbb{I}_{n-1}}{m_n} = \frac{\underline{UDU^\dagger} - \tilde{U}\underline{D}\tilde{U}^\dagger + \epsilon\, m_{n-1}\mathbb{I}_{n-1}}{m_n}, \quad |\epsilon| \leq 1. \tag{16}$$

(16) will provide the scaling behaviour for the elements of the mixing matrix that we are seeking for (see the precise examples below). For this purpose, we shall use the property that the highest mass scale occurring in the numerator of the r.h.s. of (16) is m_{n-1} and that all the matrix elements involved have, because of unitarity, moduli smaller or equal to 1. So, the modulus of the numerator is $m_{n-1}\times$ (*coefficient of modulus* $\leq 2(n-1)^2+1$) [8] . As cancellations are expected between $\underline{UDU^\dagger}$ and $\tilde{U}\underline{D}\tilde{U}^\dagger$, this bound is expected not to be saturated (see subsection 5.2 below.)

Further comments about the condition (13) are due, to show that the property 4 of section 4 makes it a very mild assumption. First, if a 3×3 mass matrix $\mathcal{M}$ ($\mathcal{M}$ is the mass matrix M normalized to the largest corresponding quark mass) is hierarchical, the eigenvalues of $\tilde{\mathcal{M}}$ differ very little from the quark masses for two generations (see the demonstration below); so, $\tilde{\mathcal{M}}$ has the right mass spectrum, which is a necessary condition for $\tilde{\mathcal{M}} \approx \underline{\mathcal{M}}$; then, as far as masses are concerned, the spectrum of fermions is the only property which can be detected in the framework of the Standard Model, in the sense that no criterion exists *a priori*, without additional assumptions concerning, e.g., flavour symmetry, which can select one among several Yukawa matrices yielding the same spectrum.

Our assumption can be rephrased as follows: in a basis where $\mathcal{M}$ is hierarchical, among all possible $\underline{\mathcal{M}}$'s which lead to the right quark spectrum for $(n-1)$ generations, the choice of the $(n-1)\times(n-1)$ upper left sub-matrix $\tilde{\mathcal{M}}$ for $\underline{\mathcal{M}}$ is considered to be the natural one. By stating that the mass matrix for n generations can be obtained by "dressing" the one for $(n-1)$ generations, we express in particular our

[7] We shall make in the rest of the paper the choice of the identity matrix for simplicity, but this choice is not mandatory.

[8] The $2(n-1)^2+1$ comes from the following: each entry in $\underline{U_d D_d U_d}^\dagger$ and in $\tilde{U_d}\underline{D_d}\tilde{U_d}^\dagger$ is the sum of $(n-1)^2$ terms, each of them being the product of a $(mass)$ and of two matrix elements of modulus smaller or equal to 1; $\mathbb{I}_2$ contributes the last 1.

view, emphasized in the introduction, that a coherent description of the physics for $(n-1)$ generations knows very little about higher generations.

Last, according to property 4 of section 4, $\mathcal{M}_u$ and $\mathcal{M}_d$ can always be chosen as hierarchical (by a common change of basis on u- and d-type quarks) if the mixing matrix K is hierarchical. Hence, in this case, our results do not depend on the basis chosen for $\mathcal{M}_u$ and $\mathcal{M}_d$.

The proposition above concerning the eigenvalues of $\tilde{\mathcal{M}}$ is easily proven. Up to an overall normalization factor, a hierarchical hermitian $\mathcal{M}$ can be written as

$$\mathcal{M} = \begin{pmatrix} a_{11} & a_{12} & \eta_1 \\ \overline{a_{12}} & a_{22} & \eta_2 \\ \overline{\eta_1} & \overline{\eta_2} & 1 \end{pmatrix}, \quad \text{with} \quad |\eta_1|, |\eta_2| \ll 1. \tag{17}$$

Its characteristic equation reads $\Delta(\lambda) = 0$ with

$$\Delta(\lambda) = (1-\lambda)\tilde{\Delta}(\lambda) + \lambda(|\eta_1|^2 + |\eta_2|^2) - a_{11}|\eta_2|^2 - a_{22}|\eta_1|^2 + a_{12}\overline{\eta_1}\eta_2 + \overline{a_{12}}\eta_1\overline{\eta_2}, \tag{18}$$

where $\tilde{\Delta}(\lambda)$ is the characteristic equation for $\tilde{\mathcal{M}}$; (17) shows that, up to order η^2, $\Delta(\lambda)$ vanishes for $\lambda = 1$ and for the eigenvalues of $\tilde{\mathcal{M}}$; the roots of $\Delta(\lambda) = 0$ build up the quark spectrum and thus, in particular, up to $\mathcal{O}(\eta^2)$, $\tilde{\mathcal{M}}$ has the right spectrum for $(n-1)$ generations.

5.2 Example 1: the case of two generations

Let us for example consider d-type quarks. M_d is a (real) symmetric 2×2 matrix and we have

$$M_d = U_d D_d U_d^T \quad \text{with} \quad D_d = \begin{pmatrix} m_d & 0 \\ 0 & m_s \end{pmatrix} \quad \text{and} \quad U_d = \begin{pmatrix} c_d & s_d \\ -s_d & c_d \end{pmatrix}. \tag{19}$$

One has $\tilde{U}_d = c_d, \quad m_n = m_s, \quad m_{n-1} = m_d, \quad \underline{M_d} = \underline{D_d} = m_d$.

$\tilde{A}$ is defined by $\widetilde{U_d D_d U_d^T} \equiv (U_d D_d U_d^T)_{11} = \tilde{U}_d \underline{D_d} \tilde{U}_d^T + m_s \tilde{A}$, and one has explicitly $\widetilde{U_d D_d U_d^T} = c_d^2 m_d + s_d^2 m_s$, $\tilde{U}_d \underline{D_d} \tilde{U}_d^T = c_d^2 m_d$. This yields $\tilde{A} = s_d^2$. The condition (15) takes the form

$$m_d = c_d^2 m_d + s_d^2 m_s - \epsilon_d\, m_d \quad \text{or} \quad s_d^2 = \frac{\epsilon_d\, m_d}{m_s - m_d}, \quad \text{with} \quad \epsilon_d \leq 1, \tag{20}$$

and, for $m_s \gg m_d$, the small $\sin\theta_d \approx \theta_d$ scales likes $\sqrt{\epsilon_d \frac{m_d}{m_s}}$ with $\epsilon_d \leq 1$. The same argument applied to u-type quarks yields: θ_u scales like $\sqrt{\epsilon_u \frac{m_u}{m_c}}$, $\epsilon_u \leq 1$ when $m_c \gg m_u$.

So, the (small) Cabibbo angle $\theta_c = \theta_d - \theta_u$ scales like

$$\theta_c \to \sqrt{\epsilon_d \frac{m_d}{m_s}} - \sqrt{\epsilon_u \frac{m_u}{m_c}}, \quad \epsilon_d, \epsilon_u \leq 1, \quad \text{when} \quad m_s, m_c \gg m_d, m_u, \tag{21}$$

which is well known [19] (see also [6] and references therein) to be compatible with the experimental values of the quark masses. This encouraging result we shall extend to the case of three generations.

5.3 Example 2: the case of three generations

One has

$$M_d = U_d D_d U_d^\dagger \quad \text{with} \quad D_d = \begin{pmatrix} m_d & 0 & 0 \\ 0 & m_s & 0 \\ 0 & 0 & m_b \end{pmatrix} \quad \text{and} \quad \underline{D_d} = \begin{pmatrix} m_d & 0 \\ 0 & m_s \end{pmatrix}. \tag{22}$$

The general condition (13) writes

$$\underline{M_d} = \underline{U_d D_d U_d}^\dagger = \widetilde{U_d D_d U_d^\dagger} - \epsilon_d\, m_s \mathbb{I}_2, \quad |\epsilon_d| \leq 1. \tag{23}$$

According to (14), the equation

$$\widetilde{U_d D_d U_d^\dagger} = \tilde{U}_d \underline{D_d} \tilde{U}_d^\dagger + m_b \begin{pmatrix} |(U_d)_{13}|^2 & (U_d)_{13}\overline{(U_d)_{23}} \\ \overline{(U_d)_{13}}(U_d)_{23} & |(U_d)_{23}|^2 \end{pmatrix} \tag{24}$$

defines $\tilde{A}$ as the 2×2 matrix factorizing m_b. (16) writes now:

$$\tilde{A} \equiv \begin{pmatrix} |(U_d)_{13}|^2 & (U_d)_{13}\overline{(U_d)_{23}} \\ \overline{(U_d)_{13}}(U_d)_{23} & |(U_d)_{23}|^2 \end{pmatrix} = \frac{\underline{\underline{U_d D_d U_d}}^\dagger - \tilde{U}_d \underline{D_d} \tilde{U_d}^\dagger + \epsilon_d\, m_s \mathbb{I}_2}{m_b}, \quad |\epsilon_d| \leq 1. \tag{25}$$

All elements of the unitary matrices U_d and $\underline{U_d}$ having moduli bounded by 1, and the highest mass scale occurring in the numerator of the l.h.s. of (25) being m_s, one gets

$$[|(U_d)_{13}|, |(U_d)_{23}|] \approx [\eta_{13}, \eta_{23}] \sqrt{\frac{m_s}{m_b}}, \tag{26}$$

where η_{13}, η_{23} are coefficients at most equal to $2(n-1)^2 + 1 \equiv 9$. The unitarity of U_d ensures that the same bounds occur for $|(U_d)_{31}|, |(U_d)_{32}|$. The same argument applied to u-type quarks entails that

$$[|(U_u)_{13}|, |(U_u)_{23}|, |(U_u)_{31}|, |(U_u)_{32}|] \approx [\zeta_{13}, \zeta_{23}, \zeta_{31}, \zeta_{32}] \sqrt{\frac{m_c}{m_t}}, \tag{27}$$

with ζ coefficients at most equal to 9. Equations (26) - (28), being direct consequences of the general requirement (13), have been obtained independently of the existence of any mass hierarchy. However, it is clear that they can be of no use when there is no hierarchy. When $m_s \ll m_b$, $m_c \ll m_t$, this proves the hierarchical property of U_u and U_d in the general sense defined in section 4. By the properties emphasized there and the definition (3), it entails that K is hierarchical according to:

$$[|K_{13}|, |K_{23}|, |K_{31}|, |K_{32}|] \approx [\beta_{13}, \beta_{23}, \beta_{31}, \beta_{32}] \sqrt{\frac{m_c}{m_t}} \pm [\delta_{13}, \delta_{23}, \delta_{31}, \delta_{32}] \sqrt{\frac{m_s}{m_b}}, \tag{28}$$

with β, δ coefficients formally not exceeding 9 and typically much smaller than 9.

6 Strengthening the bounds

By two different methods, we have obtained two different types of scaling behaviors for the Cabibbo angle. Using as input in section 3 the existence of hierarchies and the experimentally observed smallness of this angle, we deduced in (11) a scaling behaviour like the inverse of the heaviest (s and c) quarks. Using the milder “stability” requirement (13) for Yukawa matrices in section 5, and no particular input concerning either hierarchies of masses or smallness of mixing angles, we have instead deduced in (21) a behaviour like the inverse of the square root of the heaviest quark masses. This last result is certainly not optimal since very little information has been used. The combined action of large cancellations in the numerator of the r.h.s. of (25) and of a possible dependence of ϵ on the quark masses and on m_{ij} (the sole condition on it is that it is bounded by 1) can concur to strengthen the results obtained there. It is consequently instructive to consider the possibility of strengthening the bounds by incorporating our experimental knowledge on mass hierarchies and on the smallness of the mixing angles. We study

explicitly the case of three generations. It proves convenient, for the problem of interest, to use a triangular basis for the mass matrices [9].

As stated in property *(iv)* of section 2, the change of basis from hermitian to triangular can be achieved by independent right-handed rotations on u- and d-type quarks; it does not modify the mixing matrix K. Also, when hierarchies exist, the non-diagonal elements generated in the lower triangle when going from upper triangular to hermitian mass matrices differ very little from the transposed conjugates of the ones of the starting triangular matrix [10], respecting in particular the "stability" requirement [9] .

The triangular matrices for u and d-type quarks we call T_u and T_d. Their diagonalisation writes

$$T_d T_d^\dagger = U_d D_d^2 U_d^\dagger, \quad T_u T_u^\dagger = U_u D_u^2 U_u^\dagger. \tag{29}$$

Let us work, for example, in the d sector, with

$$T_d = \begin{pmatrix} m_{11} & m_{12}\, e^{i\phi_{12}} & m_{13}\, e^{i\phi_{13}} \\ 0 & m_{22} & m_{23}\, e^{i\phi_{23}} \\ 0 & 0 & m_{33} \end{pmatrix}, \quad U_d = (U_d)_{ij},\ i,j = 1\cdots 3, \quad D_d = diag(m_d, m_s, m_b), \tag{30}$$

where the m_{ij} are real coefficients. Equation (29) determines the unknown $(U_d)_{ij}, m_d, m_s, m_b$ as functions of the independent parameters m_{ij}, ϕ_{ij}. In a Wolfenstein-like parameterization of K [15] and with $\lambda \approx .22$, which includes the information that mixing angles are small, (29), (30) yield respectively for $(T_d T_d^\dagger)_{33}$, $(T_d T_d^\dagger)_{23}$ and $(T_d T_d^\dagger)_{13}$ (see [9])

$$\begin{aligned} m_{33}^2 &= m_b^2 |(U_d)_{33}|^2 \left(1 + \mathcal{O}(\lambda^8)\right), \\ m_{33} m_{23}\, e^{i\phi_{23}} &= m_b^2 \overline{(U_d)_{33}} (U_d)_{23} \left(1 + \mathcal{O}(\lambda^4)\right), \\ m_{33} m_{13}\, e^{i\phi_{13}} &= m_b^2 \overline{(U_d)_{33}} (U_d)_{13} \left(1 + \mathcal{O}(\lambda^4)\right), \end{aligned} \tag{31}$$

and one finally gets

$$\begin{aligned} m_b(m_{ij}, \phi_{ij}) &\approx \frac{|m_{33}|}{|(U_d)_{33}|(m_{ij}, \phi_{ij})} \left(1 + \mathcal{O}(\lambda^8)\right), \\ |(U_d)_{23}(m_{ij}, \phi_{ij})| &\approx \frac{|m_{23}|}{m_b(m_{ij}, \phi_{ij})} \left(1 + \mathcal{O}(\lambda^4)\right), \\ |(U_d)_{13}(m_{ij}, \phi_{ij})| &\approx \frac{|m_{13}|}{m_b(m_{ij}, \phi_{ij})} \left(1 + \mathcal{O}(\lambda^4)\right). \end{aligned} \tag{32}$$

The last two equations of (32) provide a $1/m_{b,t}$ behaviour only if the mass m_b in the denominator can be made very large independently of the non-diagonal elements m_{13} and m_{23} of the Yukawa matrix T_d. We have seen in section 3 that it is what occurs for two generations.

The authors of the first reference in [9] have shown that the inversion of the system of equations in (31) plus those associated with the other entries of $(T_d T_d^\dagger)$ yields, in particular $m_b = m_{33}(1 + \mathcal{O}(\lambda^4))$. Accordingly, the limit of very large m_b depends only on m_{33}, and not on m_{13} and m_{23}. The same argument can be made for the u-type quarks, and, (3) yields a result for the CKM mixing matrix K.

In the presence of hierarchies the u- and d- type quarks (m_b and m_t are much larger than the other quark masses) and when the mixing angles are small, the non-diagonal "external" matrix elements $|K_{13}|, |K_{23}|, |K_{31}|, |K_{32}|$ scale like

$$[|K_{13}|, |K_{23}|, |K_{31}|, |K_{32}|] \approx \frac{[\mu_{13}, \mu_{23}, \mu_{31}, \mu_{32}]}{m_b} \pm \frac{[\nu_{13}, \nu_{23}, \nu_{31}, \nu_{32}]}{m_t} \tag{33}$$

where the μ's and ν's are independent mass scales.

[9] The advantage of triangular basis [9][10] is that the diagonal elements of the mass matrices, identical, then, to their eigenvalues, are, when hierarchies exist, very close to the quark masses, and that their entries can be expressed in terms of the elements of the CKM mixing matrix and of the quark masses.

7 Conclusion

We have shown that, under a very general assumption concerning the stability of Yukawa matrices as one steps up or down generations, the non-diagonal border elements (lowest line and right column) of the CKM mixing matrix scale like the inverse square root of the largest masses of the upper generation of fermions. For two generations, one recovers a well known behaviour for the Cabibbo angle. A more constraining behaviour like the inverse of the heaviest masses instead of their square root has been obtained by using the existence of mass hierarchies and the smallness of the mixing angles.

Acknowledgements. We are indebted to M.B. Gavela, with whom this work initiated. It is a pleasure to thank B. Stech and A. Romanino for discussions and suggestions. S.T.P. acknowledges with gratefulness the hospitality of LPTHE, Université. de Paris VI, where part of the work on the present study was done. B.M. wants to thank SISSA, where this work was completed, for the kind hospitality provided to him.

References

[1] *S.L. GLASHOW: Nucl. Phys. 22 (1961) 579;
A. SALAM: in "Elementary Particle Theory: Relativistic Groups and Analyticity" (Nobel symposium No 8), edited by N. Svartholm (Almquist and Wiksell, Stockholm 1968);
S. WEINBERG: Phys. Rev. Lett. 19 (1967) 1264.*

[2] *M. OLECHOWSKI & S. POKORSKI: Phys. Lett. B 257 (1991) 388.*

[3] *P. RAMOND, R.G. ROBERTS & G.G. ROSS: Nucl. Phys. B 406 (1993) 19.*

[4] *R.D. PECCEI & K. WANG: Phys. Rev. D 53 (1996) 2712.*

[5] *H. GEORGI & C. JARLSKOG: Phys. Lett. B 86 (1979) 297.*

[6] *H. FRITZSCH: Phys. Lett. B 70 (1977) 426, B 184 (1987) 391, and B 189 (1987) 191.*

[7] *H. FRITZSCH & Zhi-zong XING: Phys. Lett. B 413 (1997) 396.*

[8] *D. FALCONE & F. TRAMONTANO: Phys. Rev. D59 (1999) 017302.*

[9] *T.K. KUO, S.W. MANSOUR & G.H. WU: Phys. Rev. D 60: (1999) 093004, and Phys. Lett. B 467 (1999) 116; S.H. CHIU, T.K. KUO & G.H. WU: Phys. Rev. D 62 (2000) 053014.*

[10] *B.R. DESAI & A. VAUCHER: "Quark Mass Matrices with Four and Five Texture Zeroes, and CKM Matrix, in terms of Mass Eigenvalues", hep-ph/000723.*

[11] *G.C. BRANCO, D. EMMANUEL-COSTA & R. GONZÁLEZ FELIPE: Phys. Lett. B477 (2000) 147.*

[12] *M.S. BERGER & K. SIYEON: "Discrete Flavor Symmetries and Mass Matrix Textures", hep-ph/0005249.*

[13] *R. BARBIERI, L.J. HALL & A. ROMANINO: Phys. Lett. B 401 (1997) 47.*

[14] *N. CABIBBO: Phys. Rev. Lett. 10 (1963) 531;
M. KOBAYASHI and T. MASKAWA: Prog. Theor. Phys. 49 (1973) 652.*

[15] *L. WOLFENSTEIN: Phys. Rev. Lett. 51 (1983) 1945.*

[16] *F.W. TURNBULL & A.C. AITKEN: "An Introduction to the Theory of Canonical Matrices" (Dover, New York, 1961) p. 194.*

[17] *P.H. FRAMPTON & C. JARLSKOG: Phys. Lett. 154 B (1985) 421.*

[18] *F. FERUGLIO, A. MASIERO & L. MAIANI: Nucl. Phys. B 387 (1992) 523;
F. FERUGLIO: "The physics of the chiral fermions", hep-ph/9405260.*

[19] *S. WEINBERG: Trans. NY. Acad. Sci. (Ser. II) 38 (1977) 185.*